KYUSHU-HET/021

Dynamics of the Light-Cone Zero Modes: Theta Vacuum of the Massive Schwinger Model

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(March 19, 1996)

Abstract

The massive Schwinger model is quantized on the light cone with great care on the bosonic zero modes by putting the system in a finite (light-cone) spatial box. The zero mode of A_{-} survives Dirac's procedure for the constrained system as a dynamical degree of freedom. After regularization and quantization, we show that the physical space condition is consistently imposed and relates the fermion Fock states to the zero mode of the gauge field. The vacuum is obtained by solving a Schrödinger equation in a periodic potential, so that the theta is understood as the Bloch momentum. We also construct a one-meson state in the fermion-antifermion sector and obtained the Schrödinger equation for it.

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I. INTRODUCTION

In a recent series of papers [1–3], we have examined the massive Schwinger model in the Light-Front Tamm-Dancoff (LFTD) approximation [4] and have obtained some interesting non-perturbative results which have never obtained by other methods, such as bosonization and lattice simulations. The power of the LFTD approximation suggests that, once it is applied to QCD_{1+3}, we would have remarkable progress in the study of relativistic bound state problems.

The LFTD approximation is the Tamm-Dancoff approximation [5] (a truncation of the infinite dimensional Fock space by limiting the number of constituents) applied to field theory quantized on the light cone. The light-cone quantization is essential for the Tamm-Dancoff approximation: the vacuum is quite simple in the light-cone quantization because pair creations (annihilations) from (into) the vacuum are kinematically suppressed.

How does a complex vacuum structure emerge in light-front field theory with the simple vacuum? Actually this is one of the most important questions in light-front field theory [6]. (In our previous papers, we completely neglected the vacuum structure.) In this paper, we will discuss the simplest non-trivial example of the vacuum structure which likely leads to observable effects, the theta vacuum of the massive Schwinger model.

The massive Schwinger model has been studied by many authors [7,8] because it shares several important features with QCD_{1+3} such as quark confinement, anomalous $U(1)_A$ breaking as well as θ-vacuum. In his seminal paper [8], Coleman showed that the vacuum angle θ can be regarded as an external constant electric field. One of his important results is that the periodicity of physics in θ is a consequence of dynamical structure of the vacuum. Namely, it comes from the fact that a pair creation of a fermion and an anti-fermion from the vacuum is energetically favorable in a background electric field stronger than a certain critical value.

How can this dynamical feature of θ be understood in the light-cone quantization with a simple vacuum? The most important result of the present paper is the demonstration

that the dynamics of the zero mode of the gauge field is responsible for it. In order to explicitly extract the zero modes, we first put the system into a finite light-cone spatial box ($x^- \in [-L, L]$) and impose the periodic boundary condition [9,10], keeping in mind that we should eventually take the limit $L \to \infty$. (We neglect the fermion zero modes at all, by choosing the anti-periodic boundary condition for the fermionic variables.) Even after fixing a gauge, the zero mode of the longitudinal component of the gauge field ($\overset{o}{A}_-$) remains to be dynamical while the other gauge components ($\tilde{A}_-$,$\overset{o}{A}_+$,$\tilde{A}_+$) do not. We show that we can impose the physical state condition (the chargeless condition) consistently at the quantum level. It relates the fermion Fock states to the zero mode of the gauge field. The theory is still invariant under large gauge transformations ($\pi_1(S^1) = Z$). We look for states which satisfy the physical state condition and are invariant under large gauge transformations. The vacuum is obtained by solving a simple Schrödinger equation in a periodic potential. The theta variable is identified as the Bloch momentum. We also obtain the Schrödinger equation for the meson state in the fermion-antifermion sector.

There are several papers on the massive Schwinger model on the light cone. Bergknoff [11] first applied the LFTD approximation and Mo and Perry [12] refined his calculations by using the method of basis functions. We have achieved six-body LFTD calculations in order to investigate two-meson and three-meson bound states [2]. Eller, Pauli and Brodsky [13] considered the discretized light-cone quantization (DLCQ) of the massive Schwinger model. The present paper is based on these. We refer the readers to them.

The theta term has not been discussed much in the light-cone context. Although there are several papers on the theta term (vacua) in the *massless* Schwinger model [14], the *massive* Schwinger model has been rarely studied. Our approach may appear similar to that of Heinzl, Krusche and Werner [15], who discussed the zero modes of the gauge field in the massless Schwinger model and how the theta vacua arises. There are, however, critical differences; (1) Their treatment of the regularized current is not adequate. In their paper the chiral anomaly is *not* derived through the point-splitting regularization but as a consequence of the *classical* equation of motion. (2) They impose a regularized charge

density as an additional constraint, which leads to a *second-class* zero-mode Gauss law (the chargeless condition). (3) The zero mode of the gauge field can take only certain discrete values. Of course these are different from the usual treatment of the regularized currents and the Gauss law. In the present paper, we regularize the current by point-splitting with a path-dependent phase factor to make it gauge invariant and examine the structure of constraints. We find that the regularization does not affect the structure of constraints. We end up with the first-class (zero-mode) Gauss law and the usual chiral anomaly which arises from a gauge-invariant regularization procedure. The zero mode of the gauge field can take any value without inconsistency.

We emphasize that our approach is *not* the one in which the gauge field is treated as an external field. The (zero mode of the) gauge field is treated as a full-fledged quantum-mechanical dynamical degree of freedom. We think that this distinguishes our work from most of the previous papers in which it is never clear whether the (zero mode of the) gauge field is an external c-number field or not. In order to keep our formulation as transparent as possible, we consider the states for the *whole* system. We can define the conserved, gauge-invariant charge, which in this case depends on the quantum-mechanical gauge field degree of freedom. By using the charge, we succeed in imposing physical state condition consistently. The physical state condition plays a crucial role in combining the zero mode and the fermion Fock states.

In Sec. II, we first examine the constraints and eliminate dependent degrees of freedom, paying attention to bosonic zero modes. We find that the zero mode of A_- and its canonically conjugate momentum remain to be dynamical. To quantize the theory, we need to regularize operators to make them well-defined. We show that we can define the conserved charge which does not affect the classical structure of constraints. Interestingly, the physical state condition relates the fermion Fock states to the zero mode of the gauge field.

In Sec. III, we investigate the physical states. We first consider the ground state. By imposing the physical state condition and that it is an eigenstate of the light-cone Hamiltonian, we can derive a Schrödinger equation (the vacuum equation) in a periodic potential, which

determines the vacuum state of the massive Schwinger model in the presence of theta. The theta can be identified as the Bloch momentum. The periodicity of theta is self-evident. We also consider the two-body eigenstate, the meson. The Schrödinger equation is derived and the so-called "continuum limit" (more properly, the thermodynamical limit) is discussed.

Sec. IV is devoted to the conclusion and discussions. We give a detailed discussion on current regularization and anomaly in Appendix.

II. LIGHT-CONE QUANTIZATION IN A BOX

A. constraints of the massive Schwinger model on the light cone

In this section we analyze the classical structure of constraints of the massive Schwinger model, including zero modes, and derive the Hamiltonian in a canonical way. In order to explicitly separate the zero modes from the non-zero modes of the bosonic variables, we put the system into a finite light-cone spatial box ($x^- \in [-L, L]$) with the periodic boundary condition [9,10]. For the fermionic variables we impose the anti-periodic boundary condition and disregard their zero-modes completely. We will discuss possible consequences of the inclusion of the fermionic zero modes in Section IV.

The Lagrangian density of the massive Schwinger model is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left[\gamma^\mu(i\partial_\mu - eA_\mu) - m\right]\psi \tag{2.1}$$

$$= \frac{1}{2}(\partial_+ \overset{o}{A}_-)^2 + \frac{1}{2}(\partial_+\tilde{A}_- - \partial_-\tilde{A}_+)^2 + \sqrt{2}(\psi_R^\dagger i\partial_+\psi_R + \psi_L^\dagger i\partial_-\psi_L)$$
$$- m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R) - e(\sqrt{2}\psi_R^\dagger\psi_R A_+ + \sqrt{2}\psi_L^\dagger\psi_L A_-), \tag{2.2}$$

where $\overset{o}{A}_\pm$ stands for the zero mode of $A_\pm$, $\overset{o}{A}_\pm = \frac{1}{2L}\int_{-L}^{L} dx^- A_\pm$, and $\tilde{A}_\pm$ the non-zero mode, $\tilde{A}_\pm \equiv A_\pm - \overset{o}{A}_\pm$. We will use similar notations hereafter. For other notations, we refer the readers to the previous paper [2].

The conjugate momenta are obtained as follows:

$$\overset{o}{\pi}{}^+ \equiv 2L \overset{o}{E}{}^+ \approx 0, \quad \tilde{\pi}^+ \equiv \tilde{E}^+ \approx 0, \tag{2.3}$$

$$\overset{o}{\pi}{}^{-} \equiv 2L \overset{o}{E}{}^{-} = 2L\partial_+ \overset{o}{A}_-, \quad \tilde{\pi}^- \equiv \tilde{E}^- = \partial_+ \tilde{A}_- - \partial_- \tilde{A}_+ \tag{2.4}$$

$$\pi_R^\dagger = i\sqrt{2}\psi_R^\dagger, \quad \pi_R \approx 0, \quad \pi_L^\dagger \approx 0, \quad \pi_L \approx 0. \tag{2.5}$$

Note that because the zero modes do not depend on x^-, it is useful to extract the factor L from the conjugate momenta, and that, for the fermionic variables, the daggered/undaggered momenta are conjugate to undaggered/daggered variables respectively, e.g., $\pi_R^\dagger \equiv \delta L/\delta(\partial_+\psi_R)$.

From these we see that the primary constraints [16] are as follows:

$$\theta_1 \equiv \overset{o}{E}{}^+, \quad \theta_2 \equiv \tilde{E}^+, \quad \theta_3 \equiv \pi_R^\dagger - i\sqrt{2}\psi_R^\dagger, \quad \theta_4 \equiv \pi_R, \quad \theta_5 \equiv \pi_L^\dagger, \quad \theta_6 \equiv \pi_L. \tag{2.6}$$

The total Hamiltonian becomes

$$\begin{aligned} H = \int_{-L}^{L} dx^- [\frac{1}{2}(\overset{o}{E}{}^-)^2 + \frac{1}{2}(\tilde{E}^-)^2 + \tilde{E}^- \partial_- \tilde{A}_+ - \sqrt{2}\psi_L^\dagger i\partial_-\psi_L \\ + m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R) + e(\sqrt{2}\psi_R^\dagger\psi_R A_+ + \sqrt{2}\psi_L^\dagger\psi_L A_-) + \sum_{i=1}^{6}\theta_i\lambda^i], \end{aligned} \tag{2.7}$$

where λ_i $(i = 1, \ldots, 6)$ are Lagrange multipliers. The consistency conditions for θ_3 and θ_4 only determine the Lagrange multipliers λ^4 and λ^3 respectively. The rest lead to further (secondary) constraints.

$$\varphi_1 \equiv \frac{1}{2L}\int_{-L}^{L} dx^- \sqrt{2}e\psi_R^\dagger\psi_R(x), \tag{2.8}$$

$$\varphi_2 \equiv \partial_-\tilde{E}^- - \sqrt{2}e(\psi_R^\dagger\psi_R(x))_\sim, \tag{2.9}$$

$$\varphi_5 \equiv i\partial_-\psi_L^\dagger + \frac{m}{\sqrt{2}}\psi_R^\dagger + e\psi_L^\dagger A_-, \tag{2.10}$$

$$\varphi_6 \equiv i\partial_-\psi_L - \frac{m}{\sqrt{2}}\psi_R - eA_-\psi_L. \tag{2.11}$$

The consistency conditions for these constraints do not lead to any further constraints. (The consistency conditions of φ_5 and φ_6 determine the multipliers λ_6 and λ_5 respectively, while those of φ_1 and φ_2 are satisfied automatically.) As usual we can arrange these constraints into first- or second-class ones. We find the following first-class constraints,

$$\theta_1 = \overset{o}{E}{}^+, \quad \theta_2 = \tilde{E}^+ \tag{2.12}$$

$$\varphi_1 = \frac{-ie}{2L}\int_{-L}^{L} dx^-(\pi_R^\dagger\psi_R(x) + \psi_R^\dagger\pi_R(x) + \pi_L^\dagger\psi_L(x) + \psi_L^\dagger\pi_L(x)), \tag{2.13}$$

$$\varphi_2 = \partial_-\tilde{E}^- + ie(\pi_R^\dagger\psi_R(x) + \psi_R^\dagger\pi_R(x) + \pi_L^\dagger\psi_L(x) + \psi_L^\dagger\pi_L(x))_\sim. \tag{2.14}$$

We choose the following gauge-fixing conditions,

$$\chi_1 \equiv \overset{o}{A}_+ \approx 0, \quad \chi_2 \equiv \tilde{A}_- \approx 0, \quad \chi_3 \equiv \tilde{E}^- + \partial_-\tilde{A}_+ \approx 0. \tag{2.15}$$

Note that the consistency of χ_2 gives the third constraint χ_3. The consistency of χ_1 and χ_3 determine the multiplier λ_1 and λ_2 respectively. Interestingly one cannot choose $\overset{o}{A}_- \approx 0$ because it does not have non-vanishing Poisson brackets with any of the first-class constraints. We end up with a single first-class constraint φ_1, the charge. We will impose it as a physical state condition after quantization,

$$\varphi_1|\mathrm{phys}\rangle = 0, \tag{2.16}$$

which eliminates charged states from the physical space.

We use second-class constraints to eliminate dependent degrees of freedom. It is easy to see that the independent variables are $\overset{o}{A}_-$, $\overset{o}{E}^-$, ψ_R and $\psi_R^\dagger$. Non-vanishing Dirac brackets [16] for these variables are calculated as

$$\{\psi_R(x^-), \psi_R^\dagger(y^-)\}_D = \frac{-i}{\sqrt{2}}\delta(x^- - y^-), \quad \{\overset{o}{A}_-, \overset{o}{E}^-\}_D = \frac{1}{2L}. \tag{2.17}$$

In terms of independent degrees of freedom, the Hamiltonian can be written as

$$P^- = P^-_{zero} + P^-_{fmass} + P^-_{current}, \tag{2.18}$$

$$P^-_{zero} = L(\overset{o}{E}^-)^2, \tag{2.19}$$

$$P^-_{fmass} = \frac{m^2}{\sqrt{2}}\int_{-L}^{L} dx^-[\psi_R^\dagger(x^-)e^{-ie\overset{o}{A}_- x^-}\frac{1}{i\partial_-}e^{ie\overset{o}{A}_- x^-}\psi_R(x^-)], \tag{2.20}$$

$$P^-_{current} = \frac{e^2}{2}\int_{-L}^{L} dx^- \tilde{j}^+(x^-)\left(\frac{1}{i\partial_-}\right)^2 \tilde{j}^+(x^-), \tag{2.21}$$

where the inverse of the derivative operator is understood as the principal value in the Fourier transforms [17]. Note that the dynamical zero modes ($\overset{o}{A}_-$, $\overset{o}{E}^-$) come into the expression in a nontrivial way. The first term P^-_{zero} is the energy of the constant electric field. The

second term P^-_{fmass} contains the interaction of the zero mode of the gauge field with the fermion, and requires a special care. It is interesting to note that only the non-zero mode of the current appears in the third term $P^-_{current}$.

B. regularization of composite operators, charge and subsidiary condition

In order to quantize the theory, we replace Dirac brackets with the corresponding ($(-i)$ times) equal-x^+ commutators. In addition, we need to regularize composite operators to make them well-defined. In two dimensions, one can eliminate all divergences by normal-ordering. In the following, we carefully define the current operators, Hamiltonian, and charge in a well-defined way so that the structure of constraints analyzed in the previous subsection is not altered by the regularization.

First of all, we have to define the "normal-ordering." For this purpose, we treat the gauge field $\overset{o}{A}_-$ (or, $q \equiv (L/\pi)e\overset{o}{A}_-$, which is nothing but the Chern-Simons term in one dimension) as an external field for a while and quantize the fermionic variables in this external field.

We Fourier expand the fermionic variable ψ_R,

$$\psi_R(x) = \frac{1}{2^{1/4}\sqrt{2L}} \sum_{n=-\infty}^{\infty} a_{n+\frac{1}{2}} e^{-i\frac{\pi}{L}(n+\frac{1}{2})x^-}. \tag{2.22}$$

From the corresponding Dirac brackets, $a_{n+\frac{1}{2}}$ is assumed to satisfy the following anti-commutation relations, $\{a_{n+\frac{1}{2}}, a^\dagger_{m+\frac{1}{2}}\} = \delta_{n,m}$, and $\{a_{n+\frac{1}{2}}, a_{m+\frac{1}{2}}\} = \{a^\dagger_{n+\frac{1}{2}}, a^\dagger_{m+\frac{1}{2}}\} = 0$. Using these operators, we define a set of reference states, so-called "N-vacua," in analogy of Dirac sea,

$$|0\rangle_N \equiv \prod_{n=-\infty}^{N-1} a^\dagger_{n+\frac{1}{2}}|0\rangle, \tag{2.23}$$

where $|0\rangle$ is the 'empty' state, i.e., $a_{n+\frac{1}{2}}|0\rangle = 0$ for any n. At this moment, N is an arbitrary integer. (The use of the "N-vacua" is rather standard in the Schwinger model in the equal-time quantization. See Refs. [18,19].)

We regularize the current by point-splitting. We define the current operator $j^\mu(x)$ in a gauge invariant way,

$$
\begin{aligned}
j^\mu = \lim_{\epsilon\to 0}\frac{1}{2}[\bar{\psi}(x+\epsilon)\gamma^\mu\psi(x)\exp\{-ie\int_x^{x+\epsilon} dx^\mu A_\mu\} \\
-\psi(x)\bar{\psi}(x-\epsilon)\gamma^\mu\exp\{+ie\int_{x-\epsilon}^{x} dx^\mu A_\mu\}],
\end{aligned}
\tag{2.24}
$$

where only $\overset{o}{A}_-$ and $\tilde{A}_+$ are non-zero. A straightforward calculation shows

$$
j^+(x) = \sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N + \frac{1}{2L}(N-q),
\tag{2.25}
$$

where

$$
\begin{aligned}
\sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N \ = \frac{1}{2L}\Big\{(\sum_{n\geq N}\sum_{m\geq N} + \sum_{n<N}\sum_{m\geq N} + \sum_{n\geq N}\sum_{m<N})a^\dagger_{n+\frac{1}{2}}a_{m+\frac{1}{2}} \\
- \sum_{n<N}\sum_{m<N} a_{m+\frac{1}{2}}a^\dagger_{n+\frac{1}{2}}\Big\}e^{i\pi(n-m)x^-/L}.
\end{aligned}
\tag{2.26}
$$

In Appendix, we discuss how to obtain the Schwinger term and the anomalous conservation law of the axial vector current.

A problem arises when we treat zero-modes with care. Because of the relation $j_5^\mu = -\epsilon^{\mu\nu}j_\nu$, the (+)-components of these two currents coincide. Naively, therefore, the charges should be the same. On the other hand, because the vector current is conserved and the axial-vector current is not conserved anomalously as well as explicitly, one would expect that the vector charge is conserved while the axial-vector charge is not. This apparent contradiction is resolved formally by thinking that the zero modes (the charges) have no direct connection with the non-zero modes. Perhaps an elaborate work on zero-modes may explain the precise relation between the zero modes and non-zero modes of the currents. At this moment, however, we take a pragmatic way and simply "adjust" the zero mode (charge) so that it satisfies desired properties. (See Appendix for the axial-vector charge.)

Because the Hamiltonian does not contain the zero modes of the currents, it is free from this ambiguity. What we should do is to regularize P^-_{fmass}, which is essentially the mass term $m\int dx^-\bar{\psi}\psi$ written in terms of the independent fields. But there is a rather surprising fact; the mass term $\bar{\psi}\psi$ is *not* invariant under charge conjugation on the light-cone. In equal-time quantization, in order to prove the charge conjugation invariance of the mass term, we use the fact that ψ_R anti-commutes with $\psi_L^\dagger$. In light-cone quantization, on the other hand, they do not anti-commute,

$$\{\psi_R(x^-), \psi_L^\dagger(y^-)\} = \frac{m}{4\pi}\sum_n \frac{1}{n+\frac{1}{2}-q} e^{-i\frac{\pi}{L}(n+\frac{1}{2})(x^- - y^-)}. \tag{2.27}$$

Therefore, if we wish to preserve charge conjugation invariance of P^- at the quantum level, we have to define it in an invariant way. The simplest way is to replace $\bar\psi\psi$ with $(\bar\psi\psi - \psi^T(\bar\psi)^T)/2$, where the superscript T stands for transpose. By using this definition in the quantum theory, we get [20]

$$\begin{aligned}
&\frac{m^2}{2\sqrt{2}}\int_{-L}^{L} dx^- \left[\psi_R^\dagger e^{-ie\overset{o}{A}_- x^-}\frac{1}{i\partial_-}e^{ie\overset{o}{A}_- x^-}\psi_R - e^{-ie\overset{o}{A}_- x^-}\left(\frac{1}{i\partial_-}e^{ie\overset{o}{A}_- x^-}\psi_R\right)\psi_R^\dagger\right] \\
&= \frac{m^2 L}{2\pi}\left\{\sum_{n\geq N}\frac{1}{n+\frac{1}{2}-q}a^\dagger_{n+\frac{1}{2}}a_{n+\frac{1}{2}} - \sum_{n<N}\frac{1}{n+\frac{1}{2}-q}a_{n+\frac{1}{2}}a^\dagger_{n+\frac{1}{2}}\right\} \\
&\quad + \frac{m^2 L}{4\pi}\left[\sum_{n<N}\frac{1}{n+\frac{1}{2}-q} - \sum_{n\geq N}\frac{1}{n+\frac{1}{2}-q}\right],
\end{aligned} \tag{2.28}$$

where the last term may be regularized by using ζ-function. It is rewritten as

$$\frac{m^2 L}{4\pi}\left[\psi(\frac{1}{2}+q-N) + \psi(\frac{1}{2}-q+N)\right] \tag{2.29}$$

after dropping q-independent infinity, where ψ is a digamma function.

We are now going to discuss a very interesting symmetry. Even after fixing a gauge, there is a residual symmetry, called “large” gauge symmetry. The theory is invariant under a large gauge transformation U,

$$U\psi_R(x)U^\dagger = e^{i\frac{\pi}{L}x^-}\psi_R(x), \tag{2.30}$$

$$U\ \overset{o}{A}_-\ U^\dagger = \overset{o}{A}_- - \frac{1}{e}\frac{\pi}{L}. \tag{2.31}$$

In terms of $a_{n+\frac{1}{2}}$ and $\hat q$, we have

$$U a_{n+\frac{1}{2}} U^\dagger = a_{n+\frac{3}{2}}, \tag{2.32}$$

$$U\hat q U^\dagger = \hat q - 1. \tag{2.33}$$

(In order to avoid possible confusions, we have used the notation $\hat q$ for the operator.) Note that this transformation does not change the gauge conditions, and the boundary conditions for ψ_R and A_-. This transformation generates an additive group Z and decreases q by one.

It is easy to prove the following transformation properties,

$$U|0\rangle_N = |0\rangle_{N+1} \tag{2.34}$$

$$U|q\rangle = |q+1\rangle \tag{2.35}$$

$$UP^- U^\dagger = P^- \tag{2.36}$$

$$\begin{aligned} U(\sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N)U^\dagger &= \sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_{N+1} \\ &= \sqrt{2} : \psi_R^\dagger(x)\psi_R(x) :_N - \frac{1}{2L}. \end{aligned} \tag{2.37}$$

At this point it is useful to introduce $M(q)$, the integer closest to q, i.e.,

$$-\frac{1}{2} < q - M(q) < \frac{1}{2}, \tag{2.38}$$

which transforms in the following way,

$$UM(\hat{q})U^\dagger = M(\hat{q}-1) = M(\hat{q}) - 1. \tag{2.39}$$

Let us define the charge operator. As we have explained, we do not require that the charge must be just the (light-cone) spatial integral of the current. In fact, it is easy to show that such a "naive" definition of the charge

$$Q_{\rm naive} \equiv 2L \overset{o}{J}{}^{+} = \sum_{n\geq N} a^\dagger_{n+\frac{1}{2}} a_{n+\frac{1}{2}} - \sum_{n<N} a_{n+\frac{1}{2}} a^\dagger_{n+\frac{1}{2}} + N - \hat{q} \tag{2.40}$$

does not commute with the Hamiltonian,

$$[Q_{\rm naive}, P^-] = -\frac{ie}{\pi} L \overset{o}{E}{}^{-}, \tag{2.41}$$

though it is invariant under a large gauge transformation.

We define the charge in the following way,

$$Q = \sum_{n\geq N} a^\dagger_{n+\frac{1}{2}} a_{n+\frac{1}{2}} - \sum_{n<N} a_{n+\frac{1}{2}} a^\dagger_{n+\frac{1}{2}} + N - M(\hat{q}). \tag{2.42}$$

Note that it is invariant under a large gauge transformation and commutes with the Hamiltonian. (The momentum operator $\overset{o}{E}{}^{-}$ generates an infinitesimal translation of the coordinate q. The integer part $M(q)$ is invariant under such a translation.)

We can now impose the physical state condition,

$$Q|\text{phys}\rangle = 0. \tag{2.43}$$

Because the charge is conserved and is invariant under large gauge transformations, this definition of physical states is gauge invariant and is consistent under (light-cone) "time" evolution.

III. PHYSICAL STATES

A. vacuum state

Let us consider a generic state for the total system,

$$\begin{aligned} |\phi\rangle &= \int_{-\infty}^{\infty} dq|q\rangle\langle q|\phi\rangle \\ &= \int_{-\infty}^{\infty} dq|q\rangle \sum_{\alpha} \phi_{\alpha}(q)|\text{Fock}(\alpha)\rangle \end{aligned} \tag{3.1}$$

where $\phi_\alpha(q)$ is the wave function for the zero mode in the q-representation, with α parameterizing fermion Fock states. The ket $|\text{Fock}(\alpha)\rangle$ is a fermion Fock state.

It is convenient to consider U as the product of two operators,

$$U = U_f \otimes U_g \tag{3.2}$$

where U_f acts only on the fermion variable, $U_f a_{n+\frac{1}{2}} U_f^\dagger = a_{n+\frac{3}{2}}$, and U_g only on $\hat{q}$, $U_g \hat{q} U_g^\dagger = \hat{q} - 1$.

The transformation property of $|\phi\rangle$ under U is easily derived:

$$\begin{aligned} U|\phi\rangle &= \int dq U_g|q\rangle \sum_{\alpha} \phi_{\alpha}(q) \left(U_f|\text{Fock}(\alpha)\rangle\right) \\ &= \int dq|q+1\rangle \sum_{\alpha} \phi_{\alpha}(q) \left(U_f|\text{Fock}(\alpha)\rangle\right) \\ &= \int dq|q\rangle \sum_{\alpha} \phi_{\alpha}(q-1) \left(U_f|\text{Fock}(\alpha)\rangle\right). \end{aligned} \tag{3.3}$$

Keeping this in mind, one may consider the transformation of the c-number q under the U transformation, $q \to q - 1$.

Let us now consider the ground state. We first notice that the state $|0\rangle_N$ has a smaller energy than that of any of the states of the form $\prod_{\{n_i\}} a^\dagger_{n_i+\frac{1}{2}} \prod_{\{m_i\}} a_{m_i+\frac{1}{2}}|0\rangle_N$, where $n_i \geq N$ and $m_i \leq N-1$. The problem is that the state $|0\rangle_N$ is not a physical state nor U-invariant. We therefore consider a linear combination of N-vacua and seek for the conditions under which it satisfies all the desired properties. Note that the simplification that the ground state is a linear combination of N-vacua even in the *massive* Schwinger model comes from the fact that the Hamiltonian causes no pair creation from the state $|0\rangle_N$. In equal-time quantization, on the other hand, this cannot occur.

Consider the state $|\ \rangle$,

$$|\ \rangle = \int dq|q\rangle \sum_{N=-\infty}^{\infty} \psi_N(q)|0\rangle_N, \tag{3.4}$$

and require that $Q|\ \rangle = 0$,

$$Q|\ \rangle = \int dq|q\rangle \sum_{N=-\infty}^{\infty} \psi_N(q)(N - M(q))|0\rangle_N = 0. \tag{3.5}$$

This is satisfied when $\psi_N(q) = \varphi_N(q)\delta_{N,M(q)}$ for all integer N. It appears that we may have infinitely many different functions $\varphi_N(q)$ for different values of N. But, because of the delta, each function $\varphi_N(q)$ is defined only in the region $N - 1/2 < q < N + 1/2$. As a whole, we define a single function for the whole q region, $-\infty < q < \infty$. Let us call it $\varphi(q)$. The only assumption we make is that it is a continuous function of q. By using it, the state $|\ \rangle$ can now be written as

$$\begin{aligned} |\ \rangle &= \int dq|q\rangle\varphi(q)|0\rangle_{M(q)} \\ &= \sum_{M=-\infty}^{\infty} \int_{M-\frac{1}{2}}^{M+\frac{1}{2}} dq|q\rangle\varphi(q)|0\rangle_M. \end{aligned} \tag{3.6}$$

We now require that it is an eigenstate of P^-, $P^-|\ \rangle = 2L\epsilon|\ \rangle$, where ϵ is the energy density. In the unit of $e/\sqrt{\pi} = 1$, the eigenvalue equation becomes

$$\left[-\frac{1}{2}\frac{d^2}{dq^2} + \frac{m^2}{2}\{\psi(\frac{1}{2} + q - M(q)) + \psi(\frac{1}{2} - q + M(q))\}\right]\varphi(q) = \tilde{\epsilon}\varphi(q), \tag{3.7}$$

where $\tilde{\epsilon} \equiv 4\pi\epsilon$. It is a Schrödinger equation in a periodic potential. We call this Schrödinger equation (3.7) the vacuum equation.

It is well-known as Bloch theorem [21] that the solution $\varphi(q)$ of a Schrödinger equation for a periodic potential (with period 1) can be written in the following form,

$$\varphi_\theta(q) = e^{-i\theta q}\phi(q) \tag{3.8}$$

where $\phi(q)$ is a periodic function. The parameter θ has been introduced as a Bloch momentum. The periodicity of θ is evident.

It is not difficult to see that the above θ is the vacuum angle. Consider the vacuum state,

$$\begin{aligned} |\theta\rangle &= \int_{-\infty}^{\infty} dq \varphi_\theta(q)|q\rangle|0\rangle_{M(q)} \\ &= \sum_{M=-\infty}^{\infty} \int_{M-\frac{1}{2}}^{M+\frac{1}{2}} dq \varphi_\theta(q)|q\rangle|0\rangle_M \end{aligned} \tag{3.9}$$

where the wave function $\varphi_\theta(q)$ is the eigenfunction of the vacuum equation (3.7) corresponding to the lowest eigenvalue $\tilde{\epsilon}_0(\theta)$. It is the eigenstate of U,

$$\begin{aligned} U|\theta\rangle &= \sum_{M=-\infty}^{\infty} \int_{M-\frac{1}{2}}^{M+\frac{1}{2}} dq \varphi_\theta(q) U_g|q\rangle U_f|0\rangle_M \\ &= \sum_{M=-\infty}^{\infty} \int_{M-\frac{1}{2}}^{M+\frac{1}{2}} dq \varphi_\theta(q)|q+1\rangle|0\rangle_{M+1} \\ &= \sum_{M=-\infty}^{\infty} \int_{M-\frac{1}{2}}^{M+\frac{1}{2}} dq \varphi_\theta(q-1)|q\rangle|0\rangle_M \\ &= e^{i\theta}|\theta\rangle, \end{aligned} \tag{3.10}$$

where we have used eq. (3.8) in the last step. The fact that an eigenstate of P^- is also an eigenstate of U is a direct consequence of (2.36).

Note that, though it is known that theta *is analogous to* a Bloch momentum [22], what we have shown is that theta *is* a Bloch momentum in the massive Schwinger model, with the explicit periodic potential.

In order to satisfy the requirement $P^-|\theta\rangle = 0$, one has to renormalize the energy,

$$P^- \to P^-_\theta = P^- - \frac{L}{2\pi}\tilde{\epsilon}_0(\theta). \tag{3.11}$$

This is essential to investigate the meson, which we are now going to discuss.

B. meson state

In this section, we proceed to consider the meson state, by approximating it as a two-body state. For this purpose, let us first discuss the momentum operator P^+. Naively, the momentum operator is defined as

$$
\begin{aligned}
P^+_{\text{naive}} &= \sqrt{2}\int dx^- \psi_R^\dagger i\partial_- \psi_R \\
&= \frac{\pi}{L}\left\{\sum_{n\geq N}(n+\frac{1}{2})a^\dagger_{n+\frac{1}{2}}a_{n+\frac{1}{2}} - \sum_{n<N}(n+\frac{1}{2})a_{n+\frac{1}{2}}a^\dagger_{n+\frac{1}{2}} + \sum_{n<N}(n+\frac{1}{2})\right\} \\
&= \frac{\pi}{L}\left\{\sum_{n\geq N}(n+\frac{1}{2})a^\dagger_{n+\frac{1}{2}}a_{n+\frac{1}{2}} - \sum_{n<N}(n+\frac{1}{2})a_{n+\frac{1}{2}}a^\dagger_{n+\frac{1}{2}} + \frac{N^2}{2} - \frac{1}{24}\right\},
\end{aligned}
\tag{3.12}
$$

where we have used the formula $\sum_{n<N}(n+\frac{1}{2}) = -\zeta(-1,\frac{1}{2}-N)$. It is interesting to note that, although the above expression is normal-ordered with respect to $|0\rangle_N$, the expression normal-ordered with respect to $|0\rangle_{N+1}$ has the same form (with N replaced by $N+1$.)

What we want to define is the momentum operator which satisfies (i) $[P^+, P^-_\theta] = 0$, (ii) $[P^+, Q] = 0$ and (iii) $UP^+U^\dagger = P^+$. The naive operator (3.12) satisfies the first two requirements, but does not the third;

$$
UP^+_{\text{naive}}U^\dagger = P^+_{\text{naive}} - \frac{\pi}{L}\left(Q + M(\hat{q}) - \frac{1}{2}\right). \tag{3.13}
$$

Is it possible to define such a P^+ by amending P^+_{naive}? It is not difficult to see the momentum operator P^+ defined by

$$
P^+ = \frac{\pi}{L}\left\{\sum_{n\geq N}(n+\frac{1}{2})a^\dagger_{n+\frac{1}{2}}a_{n+\frac{1}{2}} - \sum_{n<N}(n+\frac{1}{2})a_{n+\frac{1}{2}}a^\dagger_{n+\frac{1}{2}} - \frac{1}{2}(N - M(\hat{q}))^2 - M(\hat{q})Q\right\} \tag{3.14}
$$

satisfies the third requirement,

$$
UP^+U^\dagger = P^+, \tag{3.15}
$$

without failing to satisfy the first two requirements. We have already renormalized it so that $P^+|\theta\rangle = 0$.

It is useful to introduce $b_N^\dagger(n+\frac{1}{2})$ and $d_N^\dagger(n+\frac{1}{2})$ as

$$
\begin{aligned}
b_N^\dagger(n+\frac{1}{2}) &\equiv a^\dagger_{N+n+\frac{1}{2}} \\
d_N^\dagger(n+\frac{1}{2}) &\equiv a_{N-n-\frac{1}{2}},
\end{aligned}
\tag{3.16}
$$

where n is a positive integer. They act as creation operators of a fermion and an anti-fermion with respect to the N-vacuum respectively. It is easy to see that they satisfy the anti-commutation relation, $\{b_N^\dagger(n+\frac{1}{2}), b_N(n'+\frac{1}{2})\} = \{d_N^\dagger(n+\frac{1}{2}), d_N(n'+\frac{1}{2})\} = \delta_{n,n'}$.

In terms of these new operators, ψ_R, Q, and P^+ are rewritten as

$$
\psi_R(x) = \frac{e^{-i\frac{\pi}{L}Nx^-}}{2^{1/4}\sqrt{2L}} \sum_{n\geq 0} \left\{ b_N(n+\frac{1}{2}) e^{-i\frac{\pi}{L}(n+\frac{1}{2})x^-} + d_N^\dagger(n+\frac{1}{2}) e^{i\frac{\pi}{L}(n+\frac{1}{2})x^-} \right\}, \tag{3.17}
$$

$$
Q = \sum_{n\geq 0} \left[b_N^\dagger(n+\frac{1}{2}) b_N(n+\frac{1}{2}) - d_N^\dagger(n+\frac{1}{2}) d_N(n+\frac{1}{2}) \right] + N - M(\hat{q}), \tag{3.18}
$$

$$
\begin{aligned}
P^+ = {} & \frac{\pi}{L} \sum_{n\geq 0} (n+\frac{1}{2}) \left[b_N^\dagger(n+\frac{1}{2}) b_N(n+\frac{1}{2}) + d_N^\dagger(n+\frac{1}{2}) d_N(n+\frac{1}{2}) \right] \\
& + \frac{\pi}{L} \left(-\frac{1}{2}(N - M(\hat{q}))^2 + (N - M(\hat{q}))Q \right).
\end{aligned}
\tag{3.19}
$$

We are now ready to present the two-body approximation of the meson state,

$$
|K\rangle = \sum_{M=-\infty}^{\infty} \int_{M-\frac{1}{2}}^{M+\frac{1}{2}} dq |q\rangle \sum_{k=0}^{K-1} \varphi_K(q,k) b_M^\dagger(k+\frac{1}{2}) d_M^\dagger(K-k-\frac{1}{2}) |0\rangle_M, \tag{3.20}
$$

where $\varphi_K(q-1,k) = e^{i\theta}\varphi_K(q,k)$. This state is physical and an eigenstate of P^+ with eigenvalue $(\pi/L)K$.

In order to obtain the theta dependence of the mass of the meson, we need to solve the Einstein-Schrödinger equation $2P^+P^-_\theta|K\rangle = M^2(\theta)|K\rangle$. In the two-body sector in which we are working, the Einstein-Schrödinger equation becomes

$$
\begin{aligned}
& K \left[-\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{m^2}{2}[\psi(\frac{1}{2}+q-M(q)) + \psi(\frac{1}{2}-q+M(q))] - \tilde{\epsilon}_0(\theta) \right] \varphi_K(q,k) \\
& + \left[m^2 K \left(\frac{1}{k+\frac{1}{2}-q+M(q)} + \frac{1}{K-k-\frac{1}{2}+q-M(q)} \right) + \sum_{l=0,(l\neq k)}^{K-1} \frac{K}{(l-k)^2} \right] \varphi_K(q,k) \\
& + \sum_{l=0}^{K-1} \left(\frac{1}{K} - \frac{K(1-\delta_{l,k})}{(l-k)^2} \right) \varphi_K(q,l) \\
= {} & M^2(\theta)\varphi_K(q,k).
\end{aligned}
\tag{3.21}
$$

We call this equation the meson equation.

It is instructive to consider the so-called "continuum" limit, $K \to \infty$, $L \to \infty$, keeping the ratio $P^+ = (\pi/L)K$ finite, of the meson equation. Naively, the second term of (3.21) becomes independent of $c \equiv q - M(q)$,

$$K\left(\frac{1}{n+\frac{1}{2}-c} + \frac{1}{K-n-\frac{1}{2}+c}\right) \to \frac{1}{x} + \frac{1}{1-x} \tag{3.22}$$

as K goes to infinity with $n/K \to x$. (Remember $|c| < 1/2$.) Therefore it might appear that the zero mode decouples from the non-zero modes,

$$\begin{aligned} K&\left\{-\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{m^2}{2}[\psi(\frac{1}{2}+q-M(q)) + \psi(\frac{1}{2}-q+M(q))] - \tilde{\epsilon}_0\right\}\phi(q,x) \\ &+(m^2-1)\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi(q,x) + \int_0^1 dy(1-\frac{1}{(x-y)^2})\phi(q,y) = M^2\phi(q,x), \end{aligned} \tag{3.23}$$

so that the solution is the product of the solution of the vacuum equation $\varphi_\theta(q)$ and that of 'tHooft-Bergknoff equation [11,23],

$$(m^2-1)\left(\frac{1}{x}+\frac{1}{1-x}\right)\Phi(x) + \int_0^1 dy(1-\frac{1}{(x-y)^2})\Phi(y) = M^2\Phi(x). \tag{3.24}$$

There is however a subtlety; when $n = 0$ and $n = K-1$, the second term of (3.21) is divergent as $c \to 1/2$ and $c \to -1/2$, respectively. It is therefore not obvious whether the zero mode decouples or not. At this moment, we do not know if the zero mode really decouples.

IV. CONCLUSION

In this paper, we treated the zero mode of the gauge field very carefully by putting the system in a finite (light-cone) spatial box. We showed that $\overset{o}{A}_-$ survives Dirac's procedure. The Hamiltonian in terms of the independent degrees of freedom contains a complicated interaction term between the fermion and the zero mode. In order to quantize the model, we carefully defined the current, charge, and momentum operators, so that they satisfy the desired properties. In particular, we succeeded in constructing the charge operator which is

invariant under large gauge transformations and commutes with the Hamiltonian P^-. By using it, we were able to define the physical space.

A physical state, which annihilates the charge, is a state whose fermion Fock state component is related to the zero mode of the gauge field. As a very important example, we constructed the vacuum state. It turned out that it is a linear combination of infinitely many N-vacua, with the wave function satisfying the vacuum equation (3.7). The theta is identified with a Bloch momentum in a periodic potential. It is therefore self-evident that the energy density is periodic in theta. We proceeded to investigate the meson state by approximating it as a two-body state. We obtained the meson equation (3.21), which determine the theta dependence of the meson mass.

It is interesting to note that the potential of the vacuum equation (3.7) has singularities at q equal to half-odd integers. This singularities stem from the zeros of the Dirac operator $D_- = \partial_- + ie \overset{o}{A}_-$ for anti-periodic functions. A proper treatment of these zeros might have regularized the singularities of the potential. Unfortunately, however, we do not understand how to do it.

Numerical solutions of the vacuum equation (3.7) and the meson equation (3.21) are now under study.

ACKNOWLEDGMENTS

The authors are very glad to acknowledge the discussions with our colleagues in Kyushu University. K. H. would like to thank The Ohio-State University, Stanford Linear Accelerator Center, and Max-Planck-Institut für Kernphysik for the hospitality, where this work has been worked. He is grateful to R. Perry, S. Pinsky, D. G. Robertson, A. C. Kalloniatis, S. Brodsky, M. Völlinger, B. van de Sande, and T. Heinzl for the discussions. M. T. would like to thank S. Tanimura and M. Tachibana for the discussions. This work is supported by a Grand-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan (No.06640404).

APPENDIX A: CURRENTS AND ANOMALY

In this appendix, we discuss the regularization of the current, the Schwinger term, and chiral anomaly. In order to have a well-defined quantum theory, one must regularize the current properly so that it reproduces the well-known chiral anomaly.

The massive Schwinger model is a gauge invariant theory. One should preserve gauge invariance in any regularization. Actually it is possible. On the other hand, axial symmetry is broken anomalously at the quantum level. There is no consistent way to preserve both symmetries.

Let us begin with our Fourier expansion of the fermion field (2.22). By substituting it into the gauge invariant definition of the current (2.24), we get

$$j^+(x) = \sqrt{2} : \psi_R^\dagger \psi_R(x) :_N + \frac{1}{2L}(N-q), \tag{A1}$$

$$j^-(x) = \sqrt{2} : \psi_L^\dagger \psi_L(x) :_N - \frac{e}{2\pi}\tilde{A}^-, \tag{A2}$$

in our gauge condition. The normal-ordering is with respect to the N-vacuum. See eq. (2.26). In deriving these, we used the following properties [11],

$$_N\langle 0|\psi_R^\dagger(x+\epsilon)\psi_R(x)|0\rangle_N = \frac{-i}{2\sqrt{2}\pi}\frac{e^{i\frac{\pi}{L}N\epsilon^-}}{\epsilon^- - i0}, \tag{A3}$$

$$_N\langle 0|\psi_L^\dagger(x+\epsilon)\psi_L(x)|0\rangle_N = \frac{-i}{2\sqrt{2}\pi}\frac{e^{i\frac{\pi}{L}N\epsilon^-}}{\epsilon^+ - i0}. \tag{A4}$$

As explained in the text, we think that the zero mode (the charge) has nothing to do with the non-zero modes and "adjust" the zero mode so that it satisfies desired properties. In Sec. II B, we have constructed such a charge. We only require that the nonzero modes of the vector and axial vector currents satisfy the conservation and anomalous conservation laws respectively.

In order to calculate the divergences of the currents, we need the commutator of the current, $[\tilde{j}^+(x), \tilde{j}^+(y)]$. By a straightforward calculation, we get

$$[\tilde{j}^+(x), \tilde{j}^+(y)] = \frac{1}{(2L)^2}\Big[(\sum_{n=0}^{\infty}\exp\{-i\frac{\pi}{L}(n+\frac{1}{2})(x-y)\})^2 - (\sum_{n=0}^{\infty}\exp\{i\frac{\pi}{L}(n+\frac{1}{2})(x-y)\})^2\Big] + \cdots, \tag{A5}$$

where the ellipsis stands for the operator part which vanishes in the "continuum" limit. Note that the sums do not converge. We make them convergent by adding (or subtracting) a small imaginary part in the exponents. We get

$$[\tilde{j}^+(x), \tilde{j}^+(y)] = \frac{1}{4\pi}\lim_{\epsilon\to 0}[\frac{1}{(x-y+i\epsilon)^2} - \frac{1}{(x-y-i\epsilon)^2}] + \mathcal{O}(L^{-2})$$
$$= \frac{i}{2\pi}\delta'(x-y) + \mathcal{O}(L^{-2}). \tag{A6}$$

In this way, we can reproduce the correct Schwinger term in the "continuum" limit.

It is now easy to calculate the current divergences. By using the anomalous commutation relation (A6), one get

$$\partial_+\tilde{j}^+(x) = -i[\tilde{j}^+(x), P^-] = im(:\psi_L^\dagger\psi_R(x) - \psi_R^\dagger\psi_L(x):)_\sim + \frac{e}{2\pi}\partial_-\tilde{A}^-. \tag{A7}$$

The spatial derivative of $\tilde{j}^-(x)$ is

$$\partial_-\tilde{j}^-(x) = -im(:\psi_L^\dagger\psi_R(x) - \psi_R^\dagger\psi_L(x):)_\sim - \frac{e}{2\pi}\partial_-\tilde{A}^-. \tag{A8}$$

(This may of course be obtained from the commutator with P^+ defined in Sec. III B.) From these we get the divergences of the vector current and the axial vector current:

$$\partial_\mu\tilde{j}^\mu(x) = \partial_+\tilde{j}^+(x) + \partial_-\tilde{j}^-(x) = 0, \tag{A9}$$

$$\partial_\mu\tilde{j}_5{}^\mu(x) = \partial_+\tilde{j}^+(x) - \partial_-\tilde{j}^-(x)$$
$$= 2im(:\bar{\psi}\gamma_5\psi:)_\sim + \frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu\tilde{A}_\nu, \tag{A10}$$

where we use the relation $\gamma^\mu\gamma_5 = -\epsilon^{\mu\nu}\gamma_\nu$, $(\epsilon^{+-} = -1)$.

How about the axial charge? As explained in the text, it is formally equal to the (vector) charge. But because the axial vector current is not conserved, we expect that the axial charge is *not* conserved. In conclusion, there is no such a charge on the light-cone. Remember that the left-handed field ψ_L is not an independent field. The independent fields are ψ_R and $\overset{o}{A}_-$. It is well-known that axial-vector transformations are inconsistent on the light-cone [24], i.e., they are inconsistent with the constraint equation (2.11). What if one wants to define the axial-vector transformations only for the independent field ψ_R? Because of

$\gamma_5\psi_R = \psi_R$ it is equivalent to the usual (vector) phase transformations. One cannot define an axial-vector transformation, different from the usual (vector) phase transformation, in a self-consistent way. It means that the axial charge, which is supposed to be the generator of the transformation does not exit.

Mustaki proposed another definition of the axial-vector current which is conserved even for massive fermions [24]. Does it lead us to another definition of axial charge? Unfortunately it does not. Mustaki's conserved current is nothing but the vector current in the massive Schwinger model.

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